Correction Terms for Series

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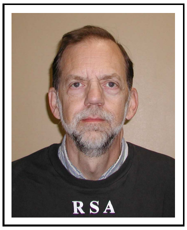

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This article uses multi-precision floating point computations to study the following three series that calculate π:

$$\pi \doteq 4 \sum_{k=0}^{N-1} \frac{(-1)^k}{2k+1} \qquad \frac{\pi^2}{12} \doteq \sum_{k=1}^{N} \frac{(-1)^{k-1}}{k^2} \qquad \frac{\pi^2}{6} \doteq \sum_{k=1}^{N} \frac{1}{k^2}$$

S1: Euler S2: Genocchi S3: Bernoulli

More specifically, this study uncovers *correction terms* that improve the accuracy of each series. These sequences of correction terms lead to the names I chose above for the three series. The results here are experimental, without mathematical proofs of their correctness, and they fall out using only arithmetic, needing no number theory or other mathematics. These results are surprising for being so easily obtained, although the same results can be found and proved using sophisticated mathematical techniques.

There is a long history of numerical experiments by mathematicians—for example, Gauss conducted extensive experiments to check his hypotheses. More recent discussions of experimental mathematics are in [1].

The basic technique was described for the series S1 in [2] and [3]. I give this description in the first section below, and then in the next two sections I carry out the same experiments for series S2 and S3.

The multi-precision calculations here were easy to program and proceeded remarkably quickly using the Python package **mpmath**, which includes a full set of transcendental functions. At the end of this article I give a short example program.

Series S1: Euler

The first number given below is π to 60 significant decimal digits. The second number is the sum of 10^7 terms of Series S1:

3.141592653589793238462643383279502884197169399375105820974944 (Pi)
3.141592**5**5358979323846**89**338327950288**1072**169399375**2**0**11334**74944 (Sum)
1 2 3 4 5 6

The series S1 is a standard example of one that converges extremely slowly—in fact for each extra digit of π one needs to add up ten times as many terms. Sure enough, with ten million terms, one only gets π accurate to 6 decimal places. But following the first incorrect digit at place 7, are 14 *correct* digits of π. (The *in*correct digits are given in **underlined boldface**.) In [2] and [3], Borwein describes this phenomenon and says it was first noticed in 1988.

One must first realize that the formula deals with so-called "real" numbers, those mysterious entities. Inside a computer, the numbers are floating point, as a partial and finite representation of some real numbers. In most computers the representation is in binary, as in **mpmath**, while in calculators the representation uses decimal numbers. In spite of this, the strange results in the table above, where there are far more correct digits of π than one would expect, come from displaying numbers in decimal form, and in adding up a number of terms that is a power of 10.

All this suggests the existence of a sequence of *correction terms* that would make the final sum much more accurate. To "correct" the first error of the sum, the erroneous digit in position 7, one needs to add in $1 \times 10^{(-7)}$. If you calculate sums for other powers of 10, you get similar results: Using $N = 10^n$ terms, the digit in position n is always one less than the true value. This leads to a possible correction term of 1×10^{-n} or $1/N$. Note that it's essential that any proposed correction term be a function of N, the number of terms added up.

Each segment of bold digits represents a potential correction term, so in the example above, there are four such segments, representing four correction terms. You always use the power of 10 of the rightmost bold digit. Notice that the fourth segment has a non-bold embedded digit, the result of chance.

Keeping in mind that $N = 10^7$, let's list the first four correction terms:

1: $(6 - 5)10^{(-7)} = \mathbf{1/N}$
2: $(64 - 89)10^{(-23)} = -25/(100N^3) = \mathbf{-1/(4N^3)}$
3: $(4197 - 1072)10^{(-39)} = 5^5/(10^4 N^5) = \mathbf{5/(4^2 N^5)}$
4: $(1058209 - 2011334)10^{(-55)} = -(61 \cdot 5^6)/(N^7 10^6) = \mathbf{-61/(4^3 N^7)}$

Next, let's look at how the first two correction terms behave with other values of N. First for $N = 500000$, each correction term still has all trailing zeros, so it only affects a few digits and nothing after that. This is why this scheme uncovers these terms. In the second example below, I chose a (sort of) random $N = 314159$. In this case the correction terms are filled with digits and affect every trailing digit of the sum, but they still produce increasingly accurate values for π.

N = 500000:

3.1415926535897932384626433832795028841971 (Pi)
3.14159**0**6535897932**40**4626433832**6**95028841972 (Original sum)

0.0000020000000000000000000000000000000000 (+1st correction)
3.1415926535897932**40**4626433832**6**9502884197**2** (Result)

0.0000000000000000020000000000000000000000 (–2nd correction)
3.14159265358979323846264338326**6**9502884197**2** (Result)

N = 314159: (Odd N, so signs of correction terms are reversed)

3.14159**583669134**3**7191**6**49824213178400**84**3524** (Original sum)

0.0000031831015504887652430775499030745577 (–1st correction)
3.14159265358979323**0399739**3**4**3**7679370097**9**46** (Result)

0.0000000000000000080629040396136839195212 (+2nd correction)
3.1415926535897932384626433833**816209293158** (Result)

1 2 3 4

It turns out that five times a power of ten works even better in calculating these correction terms, and more significant digits also helps. Using $N = 5\times10^7$, and 140 digits of accuracy, it's easy to find the 5th through the 9th terms. These numbers are too long for one line, so the first 70 digits are listed first, and below that the next 70 digits, for 140 altogether.

3.1415926535897932384626433832795028841971693993751058209749445923078164 (Pi)
3.1415926**3**35897932384626**5**38327950288419**6**16939937510582**219**4944592307 81**36** (Sum)

1 2 3 4 5 6 7

0628620899862803482534211706798214808651328230664709384460955058223172 (Pi)
3628620899862**3**8**1**3**90**2534211 70**13**9**268**480865 13**6**8**1028**6**09**09384**0731248153**2317**7** (Sum)

8 9 0 1 2 3 4

$$5:\ (640 - 363)/10^{71} = (277 \cdot 5)/(4^4 5^9 10^{63}) = \mathbf{1358/(4^4 N^9)}$$

$$6:\ (280348 - 381390)/10^{88} = -101042/(2 \cdot 2^{11} 5^{11} 10^{77}) = \mathbf{50521/(4^5 N^{11})}$$

$$7:\ (679821 - 139268)/10^{103} = 540553 \cdot 5^{13}/(2^{12} 5^{12} N^{13}) = \mathbf{2702765/(4^6 N^{13})}$$

In the same way it's easy to calculate the 8th and 9th terms from the data above.

The denominators of these terms are known as *Euler Numbers*, discovered and investigated by Euler, although the sum of the series was determined by Gregory and Leibnitz much earlier. Here are the first few correction terms and values of the first few Euler Numbers. There are equations defining the entire infinite sequence of these numbers, and an immense amount of work is associated with them.

$$
\begin{aligned}
\pi \doteq \left(4\sum_{k=0}^{N-1}\frac{(-1)^k}{2k+1}\right) &+ \frac{1}{N} + \frac{-1}{4N^3} + \frac{5}{4^2N^5} \\
&+ \frac{-61}{4^3N^7} + \frac{1385}{4^4N^9} + \frac{-50521}{4^5N^{11}} + \frac{2702765}{4^6N^{13}} \\
&+ \frac{-199360981}{4^7N^{15}} + \frac{19391512145}{4^8N^{17}} + \dots
\end{aligned}
$$

S1: Euler

Euler Numbers	
E_0	1
E_2	-1
E_4	5
E_6	-61
E_8	1385
E_{10}	-50521
E_{12}	2702765
E_{14}	-199360981
E_{16}	19391512145
...	

Series S2: Genocchi

The series S2 converges to $\pi^2/12 = 0.822467\ldots$. This number is displayed below to 150 decimal digits, and after that is the sum of 10^7 terms of the series. The incorrect digits are identified as before. Again each segment of bold digits may represent a correction term. The given sum has 11 such segments, representing the first 11 correction terms. (Sequence 10 has a correct digit of π inside it.)

$$
\begin{array}{l}
0.82246703342411321823620758332301259460947495060339921886777911468500373520160043 \\
0.8224670334241\underline{\mathbf{08}}218236\underline{\mathbf{7}}075833230125\underline{\mathbf{89}}609474950603\underline{\mathbf{54}}92188677791\underline{\mathbf{061}}8500373520\underline{\mathbf{2375}}43 \\
\qquad\qquad\quad 1 \qquad\quad 2 \qquad\quad 3 \qquad\quad 4 \qquad\quad 5 \qquad\quad 6 \qquad\quad 7 \qquad\quad 8
\end{array}
$$

$$
\begin{array}{l}
6916814450309879352652002159481168595339814362343502503893967551473165\ (\pi^2/12) \\
6916814\underline{\mathbf{34665}}9879352\underline{\mathbf{6711156}}594811\underline{\mathbf{63947494}}81436\underline{\mathbf{3784}}5\underline{\mathbf{3345}}3893\underline{\mathbf{4127250206}}65\ (\text{Sum}) \\
\qquad\qquad\quad 9 \qquad\quad 0 \qquad\quad 1 \qquad\quad 2 \qquad\quad 3 \qquad\quad 4 \qquad\quad 5
\end{array}
$$

When I first computed these correction terms, I had never heard of Genocchi Numbers, so it was exciting for me when the On-Line Encyclopedia of Integer Sequences immediately identified the sequence of numerators as these numbers. I give the correction terms at the end of this section. It's easy to determine them in the same way as in the previous section. Let's derive three of the later ones:

$$
\begin{aligned}
9&: (8595339 - 3947494)10^{(-120)} = 5 \cdot 929569/10^{120} = \mathbf{929569/(2N^{17})} \\
10&: (234350250 - 378453345)10^{(-134)} = \mathbf{-28820619/(2N^{19})} \\
11&: (9675514731 - 4127250206)10^{(-148)} = \mathbf{1109652905/(2N^{21})}
\end{aligned}
$$

Here is the series with the correction terms, along with the table of Genocchi numbers.

$$\frac{\pi^2}{12} \doteq \left(\sum_{k=1}^{N} \frac{(-1)^{k-1}}{k^2}\right) + \frac{1}{2N^2} + \frac{-1}{2N^3}$$
$$+ \frac{1}{2N^5} + \frac{-3}{2N^7} + \frac{17}{2N^9} + \frac{-155}{2N^{11}} + \frac{2073}{2N^{13}}$$
$$+ \frac{-38227}{2N^{15}} + \frac{929569}{2N^{17}} + \frac{-28820619}{2N^{19}} + \dots$$

S2: Genocchi

Genocchi Numbers	
G_0	-1
G_1	1
G_2	-3
G_3	17
G_4	-155
G_5	2073
G_6	-38227
G_7	929569
G_8	-28820619
...	

Series S3: Bernoulli

The sum of this famous series, denoted by $\zeta(2)$, was found by Euler, and since then there's been a tremendous amout of research related to the series [2]. Let's do an initial run with $N = 10^6$:

$1.6449340668482264364724151666460251892189$ ($\pi^2/6$)
$1.64493\underline{3}066848\underline{7}26436\underline{\mathbf{3057484999793918558846}}$ (sum of 10^6 terms)
1 2 3 4

Here there are 2 obvious correction terms, easily calculated as before:

1: $(4-3)/10^6 = \mathbf{1/N}$
2: $(2-7)/10^{13} = -5/(10N^2) = \mathbf{-1/(2N^2}$

The sum after adding in the two correction terms is:

$1.644933066848726436\underline{\mathbf{3057484999793918558846}}$ (sum + two terms)

But now the process seems stuck, with nothing more to do.

Digression about other bases. So far I've used a power of 10 as the number of terms, and displayed the number base 10. This works only if the correction terms have no primes in their denominators except 2 or 5. Anything else, and the term doesn't have trailing zeros. But there is nothing special about base 10. In fact, base 30 is a better choice, since it includes 2, 3, and 5. Then I try 42, then 66, and several others. Use a power of the base for the number of terms. In larger bases my Python software displays 10–35 as "a"–"z", 36–61 as "A"–"Z", and then "+-*=".

So let's do a base 30 calculation in this case, first without the two correction terms already found, and then with them included. Here $N = 30^4 = 810000$.

```
1.jad6holt3j6pimo2nsq5lfpfhms87j4ainis6smf8tga54ccc5 (π²/6, base 30)
1.jad5holtij6pdmo2nsq5mfpfhms86rleran6p312d83ebdb7a7 (sum, no corr terms)
1.jad6holt3j6pdmo2nsq5mfpfhms86rleran6p312d83ebdb7a7 (sum, 2 corr terms)

                1         2         3         4         5
```

Now there are 4 clear correction terms: the two obtained before and two new ones. Getting the terms can be annoying, since one must do base 30 arithmetic.

3: $5/30^{13} = 5/30N^3 = \mathbf{1/(6N^3)}$, since (i-d) base 30 = 5.
4: $-1/30^{21} = \mathbf{-1/30N^5}$, since (l-m) base 30 = -1.

I'm stuck again. The next base in line to try is base 42, and amazinely it works. The second line below calculates the sum of $42^3 = 74088$ terms, base 42:

```
1.r3rAvvu4s5f7hgCglndh9Cx6zwrf3b5uEFyo2msk4Drkl8rkpE (π²/6, base 42)
1.r3rAvvu4s5f7hgCglndh9Bx6zwrgk2uDfua5uxfvllg8x3flop (sum, 4 corr terms)
                1         2         3         4         5
```

This gives a fifth term.

5: $1/42^{22} = \mathbf{1/(42N^7)}$, since (C-B) base 42 = 1.

Again it stopped in base 42, so start with base 30 again, and $N = 30^3$ terms:

```
1.jad6holt3j6pimo2nsq5lfpfhms87j4ainis6smf8tga54ccc5rbtnhqgbgf (base 42)
1.jad6holt3j6pimo2nsq5lfpfhms97j4ailamn9k1277c60ghbdk4tiikas4h (4 corr terms)
                1         2         3         4         5         6
```

Now the sixth term is:

6: $-1/30^{28} = \mathbf{-1/(30N^9)}$, since (8-9) base 30 = -1.

Finally, skipping 30 and after trying 42, I try base 66, with $N = 66^3$.

```
1.GBl-HiWI6eQs*QuqzscRI8O*Qd6ikpRjScow7E07yQbhyhyYlHKT+*a6MWkb (base 42)
1.GBl-HiWI6eQs*QuqzscRI8O*Qd6ikpRjS7ow7E0ofnk7wyfvuxRnM*15ejCA (4 corr terms)
                1         2         3         4         5         6
```

The seventh term is:

7: $5/66^{34} = \mathbf{5/(66N^{11})}$, since (c-7) base 66 = 5.

I had heard of Bernoulli numbers, but I didn't remember what they were. As with the S2 Series, it was a very nice surprise to see my terms in the integer sequences encyclopedia under "Bernoulli numbers". Above, I wanted 60 base 66 digits, so I needed at least $60 \cdot \log_{10} 66 = 109.17$ or 110 decimal digits of accuracy.

Here is the final result, with three extra terms added (my approach wouldn't have been able to get the eighth term):

$$\frac{\pi^2}{6} \doteq \left(\sum_{k=1}^{N} \frac{1}{k^2}\right) + \frac{1}{N} + \frac{-1}{2\,N^2} + \frac{1}{6\,N^3}$$

$$+ \frac{-1}{30\,N^5} + \frac{1}{42\,N^7} + \frac{-1}{30\,N^9} + \frac{5}{66\,N^{11}}$$

$$+ \frac{-691}{2730\,N^{13}} + \frac{7}{6\,N^{15}} + \frac{-3617}{510\,N^{17}} + \ldots$$

S3: Bernoulli

Bernoulli Numbers	
B_0	1
B_1	-1/2 (or +1/2)
B_2	1/6
B_4	-1/30
B_6	1/42
B_8	-1/30
B_{10}	5/66
B_{12}	-691/2730
B_{14}	7/6
B_{16}	-3617/510
...	

Python Program

```
# s1.py: caculate MP value of Gregory's series
from mpmath import *   # make all parts of mpmath available
mp.dps = 63            # want 60 decimal digits precision, need a few extra
def greg(n):           # a function named "greg", parameter n
    sum = mpf(0)           # sum is a multi-precision version of 0
    sign = mpf(1)          # sign is also multi-precision 1
    for k in range(0,n):   # the range for k is 0 to n-1 inclusive
        term = sign/(k*2+1)    # term becomes multi-precision
        sign = -sign           # alternate sign
        sum += term            # running sum
    return sum             # return the sum
N = 10000000           # start of actual program
total = greg(N)*mpf(4) # get the sum and multipy by 4
print(mp.pi, "(Pi)")   # print the first line of data in this article
print(total, "(Sum)")  # print the second line of data in this article
```

References

[1] D.H. Bailey and J.M. Borwein, "Exploratory Experimentation and Computation," *Notices of the AMS*, Vol. 58, No. 10, Nov. 2011, pp. 1411–1419.

[2] Borwein, J.M., Borwein and Dilcher. "Pi, Euler Numbers, and Asymptotic Expansions," in *Pi: A Source Book*, 3rd Edition, J.L. Berggren, Borwein and Borwien, Springer, 2004, pages 642–648.

[3] Borwein, J.M., *The Life of Pi: From Archimedes to Eniac and Beyond*, online at `https://www.carma.newcastle.edu.au/jon/pi-2012.pdf`.

Summary. This article examines three series for π using a multi-precision package to calculate the sums up to 150 digits. In some cases, if the number of terms summed is a power of ten, then the sum itself displays many extra digits of π, intermingled with incorrect digits. The incorrect digits point to *correction terms* which improve the accuracy of the calculation.