THE SOFA PROBLEM

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There is an old problem in combinatorial geometry which has to do with moving furniture around corners.

PROBLEM. *What is the region of largest area which can be moved around a right-angled corner in a corridor of width one?*

Note that the problem is a two-dimensional one. The region is assumed to be rigid, and only combinations of rotations and translations are allowed.

The problem has an interesting history. It was first formally stated in print by L. Moser [**6**] in 1966, although it had been discussed informally by many people prior to 1966. The problem was also mentioned by H. T. Croft [**1**] in 1967. Croft included a rough sketch called the "Shephard piano," but the word "sofa" has now become firmly linked to the problem. In 1968, Hammersley [**3**, p. 84] gave what is currently the best-known lower bound for the maximum area: $(\pi/2)+(2/\pi)$, or approximately 2.2074. In a private communication to the author, Hammersley stated: "I had always guessed (but never been able to prove) that the lower bound above was also the exact value of [the maximum area]." Hammersley's solution (see Figure 1) can be obtained using simple calculus by considering only outer boundaries made up of two 90° arcs from a unit circle plus a variable length line segment. (The inner boundary must then be two unit segments plus a half circle.) Another lower bound was published in [**2**].

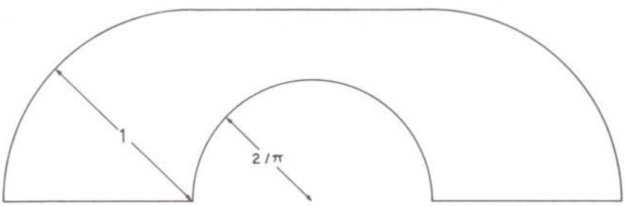

FIGURE 1.

The best known upper bound for the maximum area, $2\sqrt{2}$, was also given by Hammersley in 1968, with a proof appearing as "Editorial commentary" in 1969. (See [**3**].) Westwell [**3**] and Sebastian [**7**] claimed to have established the above upper bound, but both erroneously assumed that a bounding rectangle for the region must be rotated through a 90° angle as the corner is negotiated.

There have been several attempts to find approximate solutions to the problem. Howden [**4**] considers a general problem-solving algorithm which, to quote the referee, "comes to grief when confronted by an actual problem." Maruyama [**5**] has a much more interesting algorithm which, for a right-angled corner, produced a region of area 1.98 fairly closely resembling the solution by Hammersley. However, Maruyama does not give an error estimate or any other proof that his area is close to the maximum area.

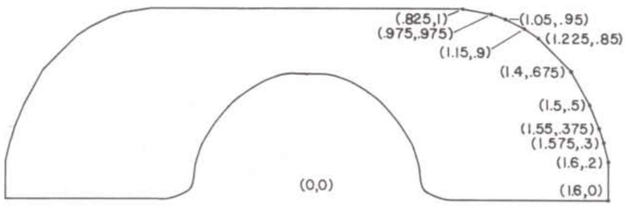

FIGURE 2.

The author independently constructed a type of Monte Carlo simulation which produced the region shown in Figure 2. This simulation might be compared to the evolution of a rigid organism which must traverse right-angled corners. All variation occurs in the outer boundary (the inner boundary is determined by turning the corner), while selection is for larger area.

Starting with widely different initial regions, the author's simulation evolved in each case to a region in close agreement with Hammersley's. The largest such region is a union of 3450 squares 1/40 units on a side, and thus with a total area of about 2.15. Figure 2 uses rectangular coordinates to specify the convex hull of the outer boundary of this region, while the inner boundary as shown is simply what would be cut out while going around the corner. The area of this figure is certainly very close to the area obtained by Hammersley. (The author has not been able to obtain an exact value for the area of Figure 2.)

It is clear that small adjustments to the outer boundary in Figure 2 will produce small improvements in the area, but the close agreement of regions obtained by entirely different methods leads one to conjecture that the areas in Figures 1 and 2 are very near to the actual maximum.

For generalizations of the sofa problem, see Maruyama [**5**], who considered general kinds of two-dimensional corridors and included shapes of various regions produced by his algorithm. Croft [**1**] included a rough sketch of a region called the "Conway Car" for reversing in a T-junction. (See also [**5**, Figure 13].) In the original problem statement [**6**], the editors noted: "For the three-dimensional version, in which the hallway has a fixed height h, it seems reasonable to suppose the answer would be a cylinder of height h whose cross-section is given by the solution of the [two-dimensional sofa problem]." Hammersley [**3**, p. 75] suggests what is perhaps a more legitimate three-dimensional version: "$\cdots$, where the passage has three straight sections each at right-angles to the other two," as in a corner in a corridor plus an elevator shaft. Concerning convex solutions to the sofa problem, the author can only remark that the maximum area is surely larger than the value of $\pi/2$ given by a semicircle.

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