

# The Life of Pi:

## From Archimedes to ENIAC and Beyond

JONATHAN M. BORWEIN, FRSC

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Canada Research Chair & Director Dalhousie Drive

**Abstract.** The desire to understand  $\pi$ , the challenge, and originally the need, to calculate ever more accurate values of  $\pi$ , the ratio of the circumference of a circle to its diameter, has challenged mathematicians—great and less great—for many many centuries and, especially recently,  $\pi$  has provided compelling examples of computational mathematics. It is also part of both mathematical culture and of the popular imagination.<sup>1</sup>

## 1 Preamble: Pi and Popular Culture

Pi, uniquely in mathematics is pervasive in popular culture. I shall intersperse this largely chronological account of Pi's mathematical status with examples of its ubiquity. More details will be found in the selected references at the end of the chapter—especially in *Pi: a Source Book* [2]. In [2] all material not otherwise referenced may be followed up upon, as may much other material, both serious and fanciful. Other interesting material is to be found in [8], which includes attractive discussions of topics such as continued fractions and elliptic integrals.

As a first example, imagine the following being written about another transcendental number:

“My name is  
Piscine Molitor Patel  
known to all as Pi Patel.

*For good measure I added*

$$\pi = 3.14$$

*and I then drew a large circle which I sliced in two with a diameter, to evoke that basic lesson of geometry.”<sup>2</sup>*

Fascination with  $\pi$  is evidenced by the many recent popular books, television shows, and movies—even perfume—that have mentioned  $\pi$ . In the 1967 *Star Trek* episode “Wolf in the Fold,” Kirk asks “*Aren't there some mathematical problems that simply can't be solved?*” And Spock ‘fries the brains’ of a rogue computer by telling it: “*Compute to the last digit the value of Pi.*” The May 6, 1993 episode of *The Simpsons* has the character Apu boast “*I can recite pi to 40,000 places. The last digit is one.*” (See Figure 1.)

In November 1996, MSNBC aired a Thanksgiving Day segment about  $\pi$ , including that scene from *Star Trek* and interviews with the present author and several other mathematicians at Simon Fraser University. The 1997 movie *Contact*, starring Jodie Foster, was based on the 1986 novel by noted astronomer Carl Sagan. In the book, the lead character searched for patterns in the digits of  $\pi$ , and after her mysterious experience found sound confirmation in the base-11 expansion of  $\pi$ . The 1997 book *The Joy of Pi* [3] has sold many thousands of copies and continues to sell well. The 1998 movie entitled *Pi* began with decimal digits of  $\pi$  displayed on the screen. Finally, in the 2003 movie *Matrix Reloaded*, the Key Maker warns that a door will be accessible for exactly 314 seconds, a number that *Time* speculated was a reference to  $\pi$ .

Equally, National Public Radio reported on April 12, 2003 that novelty automatic teller machine withdrawal slips, showing a balance of \$314,159.26, were hot in New York City. One could jot a note on the back and,

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<sup>1</sup>The *MacTutor* website, <http://www-gap.dcs.st-and.ac.uk/~history>, at the University of St. Andrews—my home town in Scotland—is rather a good accessible source for mathematical history.

<sup>2</sup>From Eli Mandel's 2002 Booker Prize winning novel *Life of Pi*.

FAX (310) 203-3852  
PHONE (310) 203-3959

A Professor at UCLA told me that  
you might be able to give me the  
answer to: What is the 40,000<sup>th</sup>  
digit of  $\pi$ ?

We would like to use the answer  
in our show. Can you help?

Figure 1: A fax from the staff of *The Simpsons* to a colleague.



Archimedes of Syracuse (287–212 BCE) was the first to show around 250 BCE that the “two possible Pi’s” are the same. Clearly for a circle of radius  $r$  and diameter  $d$ , **Area** =  $\pi_1 r^2$  while **Perimeter** =  $\pi_2 d$ , but that  $\pi_1 = \pi_2$  is not obvious. This is often overlooked.

Figure 2:  $\pi$ ’s duality

apparently innocently, let the intended target be impressed by one’s healthy saving account. Scott Simon, the host, noted the close resemblance to  $\pi$ . Likewise March 14 in North America has become  $\pi$  Day, since in the USA the month is written before the day (314). In schools throughout North America, it has become a reason for mathematics projects, especially focussing on Pi, see Figure 17. It is hard to imagine  $e$ ,  $\gamma$  or  $\log 2$  playing the same role.

For those who know *The Hitchhiker’s Guide to the Galaxy*, it is amusing that 042 occurs at the digits ending at the fifty-billionth decimal place in each of  $\pi$  and  $1/\pi$ —thereby providing an excellent answer to the ultimate question, “What is forty two?” A more intellectual offering is “The Deconstruction of Pi” given by Umberto Eco on page three of his 1988 book *Foucault’s Pendulum*, [2, p. 658].

**Pi.** Our central character

$$\pi = 3.14159265358979323\dots$$

is traditionally defined in terms of the area or perimeter of a unit circle, see Figure 2. A more formal modern definition of  $\pi$  uses the first positive zero of  $\sin$  defined as a power series. A thousand digits are recorded in Figure 20. The notation of  $\pi$  was introduced by William Jones in 1737, replacing ‘p’ and the like, and was popularized by Leonhard Euler who is responsible for much modern nomenclature.

**Why  $\pi$  is not  $22/7$ .** Despite rumours to the contrary,  $\pi$  is not equal to  $22/7$ —even the computer algebra systems *Maple* or *Mathematica* ‘know’ this since

$$(1) \quad 0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2} dx = \frac{22}{7} - \pi,$$

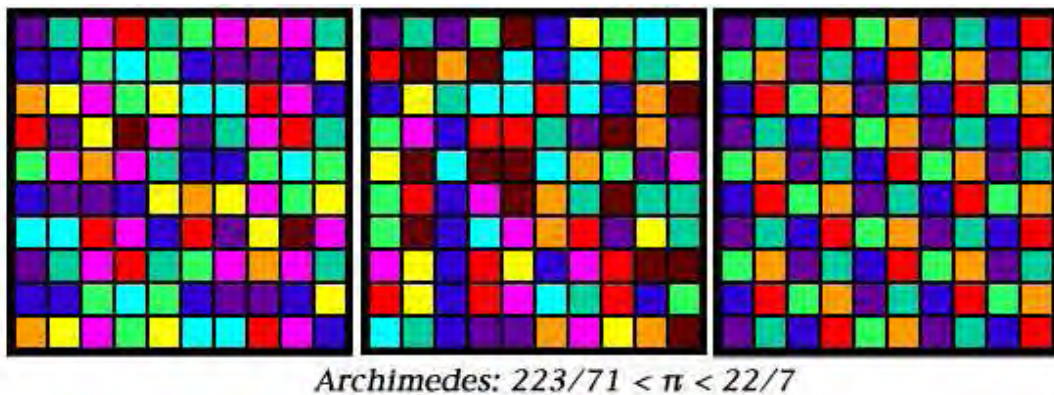


Figure 3: **A pictorial proof**

though it would be prudent to ask ‘why’ each can perform the integral and ‘whether’ to trust it?

*Assuming we trust it*, then the integrand is strictly positive on  $(0, 1)$ , and the answer in (1) is an area and so strictly positive, despite millennia of claims that  $\pi$  is  $22/7$ . In this case, requesting the indefinite integral provides immediate reassurance. We obtain

$$\int_0^t \frac{x^4(1-x)^4}{1+x^2} dx = \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4 \arctan(t),$$

as differentiation easily confirms, and so the Newtonian Fundamental theorem of calculus proves (1).

Of course  $22/7$  is one of the early continued fraction approximations to  $\pi$ . The first six convergents are

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}.$$

The convergents are necessarily good rational approximations to  $\pi$ . The sixth differs from  $\pi$  by only  $3.31 \cdot 10^{-10}$ . The corresponding simple continued fraction starts

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, \dots],$$

using the standard concise notation. This continued fraction is still very poorly understood. Compare that for  $e$  which starts

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, 1, 1, 16, 1, 1, 18, \dots].$$

Proof of this shows that  $e$  is not a quadratic irrational since such numbers have eventually periodic continued fractions.

One can take the idea in Equation (1) a bit further, as in [7]. Note that

$$(2) \quad \int_0^1 x^4(1-x)^4 dx = \frac{1}{630},$$

and we observe that

$$(3) \quad \frac{1}{2} \int_0^1 x^4(1-x)^4 dx < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2} dx < \int_0^1 x^4(1-x)^4 dx.$$

Combine this with (1) and (2) to derive:  $223/71 < 22/7 - 1/630 < \pi < 22/7 - 1/1260 < 22/7$  and so re-obtain Archimedes’ famous computation

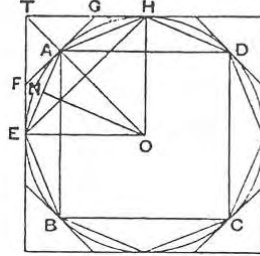
$$(4) \quad 3\frac{10}{71} < \pi < 3\frac{10}{70}.$$

# MEASUREMENT OF A CIRCLE.

## Proposition 1.

*The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.*

Let  $ABCD$  be the given circle,  $K$  the triangle described.



Archimedes' construction for the uniqueness of  $\pi$ , taken from his *Measurement of a Circle*

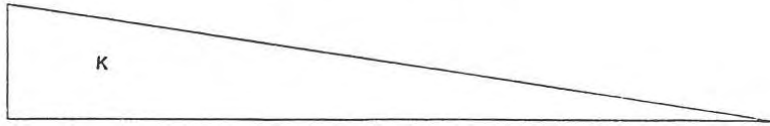


Figure 4:  $\pi$ 's Uniqueness

Figure 3 shows the estimate graphically, with the digits coloured modulo ten; one sees structure in  $22/7$ , less obviously in  $223/71$ , and not in  $\pi$ . The derivation above seems first to have been published in *Eureka*, a Cambridge student journal in 1971.<sup>3</sup>

## 2 The Childhood of Pi

Four thousand years ago, the Babylonians used the approximation  $3\frac{1}{8} = 3.125$ . Then or earlier, according to ancient papyri, Egyptians assumed a circle with diameter nine has the same area as a square of side eight, which implies  $\pi = 256/81 = 3.1604\dots$  Some have argued that the ancient Hebrews were satisfied with  $\pi = 3$ :

*“Also, he made a molten sea of ten cubits<sup>4</sup> from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about.”* (I Kings 7:23; see also 2 Chronicles 4:2)

In Judaism's defense, several millennia later, the great Rabbi Moses ben Maimon Maimonides (1135–1204) is translated by Langermann, in “The ‘true perplexity’ [2, p. 753] as fairly clearly asserting the Pi's irrationality.

*“You ought to know that the ratio of the diameter of the circle to its circumference is unknown, nor will it ever be possible to express it precisely. This is not due to any shortcoming of knowledge on our part, as the ignorant think. Rather, this matter is unknown due to its nature, and its discovery will never be attained.”* (Maimonides)

<sup>3</sup>Equation (1) was on a Sydney University examination paper in the early sixties.

<sup>4</sup>One should know that the cubit was a personal not universal measurement.

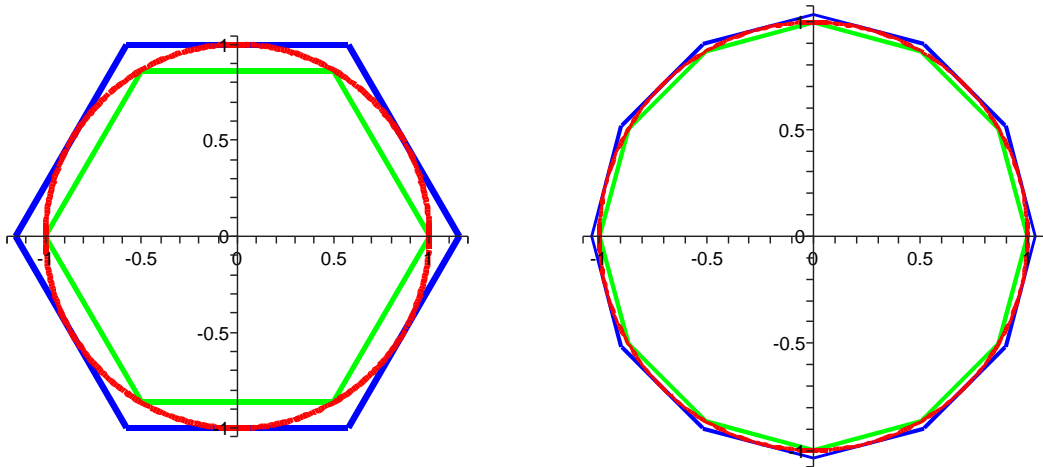


Figure 5: Archimedes' method of computing  $\pi$  with 6- and 12-gons

In each case the interest of the civilization in  $\pi$  was primarily in the practical needs of engineering, astronomy, water management and the like. With the Greeks, interest was metaphysical and geometric.

**Archimedes' Method.** The first rigorous mathematical calculation of  $\pi$  was due to Archimedes, who used a brilliant scheme based on **doubling inscribed and circumscribed polygons**

$$6 \mapsto 12 \mapsto 24 \mapsto 48 \mapsto 96$$

and computing the perimeters to obtain the bounds  $3\frac{10}{71} < \pi < 3\frac{1}{7}$ , that we have recaptured above. The case of 6-gons and 12-gons is shown in Figure 5, for  $n = 48$  one already 'sees' near-circles. Arguably no mathematics approached this level of rigour again until the 19th century.

Archimedes' scheme constitutes the first true algorithm for  $\pi$ , in that it is capable of producing an arbitrarily accurate value for  $\pi$ . It also represents the birth of numerical and error analysis—all without positional notation or modern trigonometry. As discovered severally in the 19th century, this scheme can be stated as a simple, numerically stable, recursion, as follows [5].

**Archimedean Mean Iteration (Pfaff-Borchardt-Schwab).** Set  $a_0 = 2\sqrt{3}$  and  $b_0 = 3$ . Then define

$$(5) \quad a_{n+1} = \frac{2a_nb_n}{a_n + b_n} \quad (H) \quad b_{n+1} = \sqrt{a_{n+1}b_n} \quad (G).$$

This converges to  $\pi$ , with the error decreasing by a factor of four with each iteration. In this case the error is easy to estimate, the limit somewhat less accessible but still reasonably easy [7, 5].

Variations of Archimedes' geometrical scheme were the basis for all high-accuracy calculations of  $\pi$  for the next 1800 years—well beyond its 'best before' date. For example, in fifth century CE China, Tsu Chung-Chih used a variation of this method to get  $\pi$  correct to seven digits. A millennium later, Al-Kashi in Samarkand "*who could calculate as eagles can fly*" obtained  $2\pi$  in *sexagesimal*:

$$2\pi \approx 6 + \frac{16}{60^1} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{01}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9},$$

good to 16 decimal places (using  $3 \cdot 2^{28}$ -gons). This is a personal favourite, reentering it in my computer centuries later and getting the predicted answer gave me goose-bumps.

Name	Year	Digits
Babylonians	2000? BCE	1
Egyptians	2000? BCE	1
Hebrews (1 Kings 7:23)	550? BCE	1
Archimedes	250? BCE	3
Ptolemy	150	3
Liu Hui	263	5
Tsu Ch'ung Chi	480?	7
Al-Kashi	1429	14
Romanus	1593	15
van Ceulen ( <b>Ludolph's number</b> *)	1615	35

Figure 6: **Pre-calculus  $\pi$  Calculations**

### 3 Pre-calculus Era $\pi$ Calculations

In Figures 6, 8, and 12 we chronicle the computational records during the indicated period, only commenting on signal entries.

Little progress was made in Europe during the ‘dark ages’, but a significant advance arose in India (450 CE): *modern positional, zero-based decimal arithmetic*—the “Indo-Arabic” system. This greatly enhanced arithmetic in general, and computing  $\pi$  in particular. The Indo-Arabic system arrived with the Moors in Europe around 1000 CE. Resistance ranged from accountants feared losing their livelihood to clerics who saw the system as ‘diabolical’—they incorrectly assumed its origin was Islamic. European commerce resisted into the 18th century, and even in scientific circles usage was limited until the 17th century.

The prior difficulty of doing arithmetic<sup>5</sup> is indicated by college placement advice given a wealthy German merchant in the 16th century:

*“A wealthy (15th Century) German merchant, seeking to provide his son with a good business education, consulted a learned man as to which European institution offered the best training. ‘If you only want him to be able to cope with addition and subtraction,’ the expert replied, ‘then any French or German university will do. But if you are intent on your son going on to multiplication and division—assuming that he has sufficient gifts—then you will have to send him to Italy.’”* (George Ifrah, [7])

**Ludolph van Ceulen (1540-1610).** The last great Archimedean calculation, performed by van Ceulen using  $2^{62}$ -gons—to 39 places with 35 correct—was published posthumously. The number is still called Ludolph’s number in parts of Europe and was inscribed on his head-stone. This head-stone disappeared centuries ago but was rebuilt, in part from surviving descriptions, recently as shown in Figure 7. It was reconsecrated on July 5th 2000 with Dutch royalty in attendance. Ludolph van Ceulen, a very serious mathematician, was also the discoverer of the cosine formula.

### 4 Pi’s Adolescence

The dawn of modern mathematics appears in *Vieta’s* or *Viète’s product* (1579)

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2+\sqrt{2}}}{2} \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \dots$$

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<sup>5</sup>Claude Shannon had a mechanical calculator called *Throback 1* built to compute in Roman, at Bell Labs in 1953.



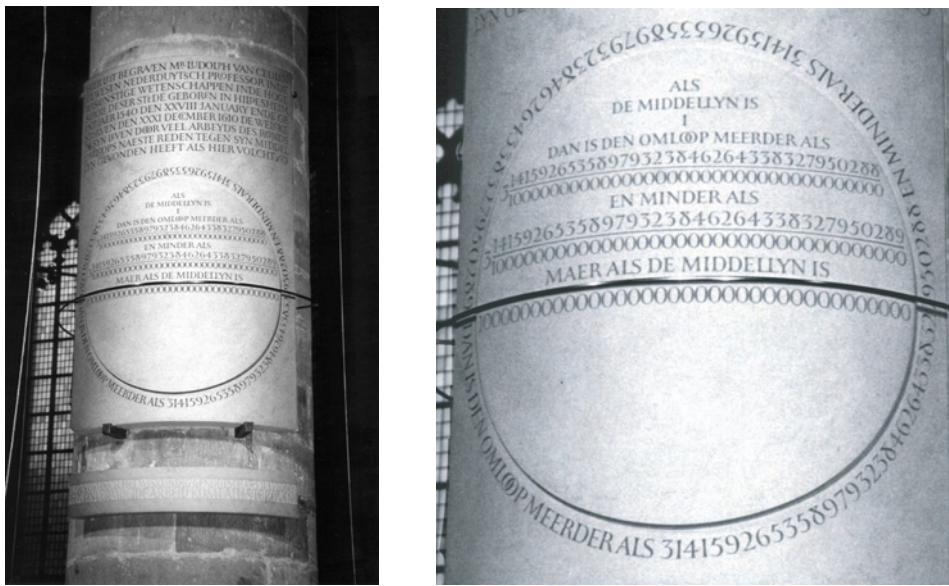


Figure 7: Ludolph's rebuilt tombstone in Leiden

considered to be the first truly infinite product; and in the *first infinite continued fraction* for  $2/\pi$  given by Lord Brouncker (1620-1684), first President of the Royal Society of London:

$$\frac{2}{\pi} = \frac{1}{1 + \frac{9}{2 + \frac{25}{2 + \frac{49}{2 + \dots}}}}.$$

This was based on the following brilliantly ‘interpolated’ product of John Wallis<sup>6</sup> (1616-1703)

$$(6) \quad \prod_{k=1}^{\infty} \frac{4k^2 - 1}{4k^2} = \frac{2}{\pi},$$

which led to the discovery of the Gamma function, see below, and a great deal more.

Equation (6) may be derived from Euler’s (1707-1783) product formula for  $\pi$ , given below in (7), with  $x = 1/2$ , or by repeatedly integrating  $\int_0^{\pi/2} \sin^{2n}(t) dt$  by parts. One may divine (7) as Euler did by *considering*  $\sin(\pi x)$  *as an ‘infinite’ polynomial* and obtaining a product in terms of the roots— $0, \{1/n^2 : n = \pm 1, \pm 2, \dots\}$ . It is thus plausible that

$$(7) \quad \frac{\sin(\pi x)}{x} = c \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

Euler, full well knowing the whole argument was heuristic, argued that, as with a polynomial,  $c$  was the value at zero, 1, and the coefficient of  $x^2$  in the Taylor series must be the sum of the roots. Hence,

$$\sum_n \frac{1}{n^2} = \frac{\pi^2}{6}.$$

<sup>6</sup>One of the few mathematicians whom Newton admitted respecting, and also a calculating prodigy!

This also leads to the evaluation of  $\zeta(2n)$  as a rational multiple of  $\pi^{2n}$ :

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \zeta(8) = \frac{\pi^8}{9450}, \dots$$

in terms of the *Bernoulli numbers*,  $B_n$ , where  $t/(\exp(t) - 1) = \sum_{n \geq 0} B_n t^n / n!$ , gives a generating function for the  $B_n$  which are perforce rational. The explicit formula<sup>7</sup>

$$\zeta(2m) = (-1)^{m-1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m}.$$

Much less is known about odd integer values of  $\zeta$ , though they are almost certainly not rational multiple of powers of  $\pi$ .

Two centuries later, in 1976 Apéry, [2, p. 439], showed  $\zeta(3)$  to be irrational, and we now also can prove that *at least one of*  $\zeta(5), \zeta(7), \zeta(9)$  or  $\zeta(11)$  is irrational, but we can not guarantee which one. All positive integer values are strongly believed to be irrational.

**More about Gamma.** One may define

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

for  $\operatorname{Re} x > 0$ . The starting point is that

$$(8) \quad x \Gamma(x) = \Gamma(x+1), \quad \Gamma(1) = 1.$$

In particular, for integer  $n$ ,  $\Gamma(n+1) = n!$ . Also for  $0 < x < 1$

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

since for  $x > 0$  we have

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n! n^x}{\prod_{k=0}^n (x+k)}.$$

This is a nice consequence of the Bohr-Mollerup theorem which shows that  $\Gamma$  is the unique log-concave function on the positive half line satisfying (8). Hence,  $\Gamma(1/2) = \sqrt{\pi}$  and equivalently we evaluate the *Gaussian integral*

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi},$$

so central to probability theory. In the same vein, the improper *sinc* function integral evaluates as

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \pi.$$

Considerable information about the relationship between  $\Gamma$  and  $\pi$  is to be found in [7, 8].

**François Vieta (1540-1603).** A flavour of Vieta's writings can be gleaned in this quote from his work, first given in English in [2, p. 759].

*“ Arithmetic is absolutely as much science as geometry [is]. Rational magnitudes are conveniently designated by rational numbers, and irrational [magnitudes] by irrational [numbers]. If someone measures magnitudes with numbers and by his calculation get them different from what they really are, it is not the reckoning's fault but the reckoner's.*

*Rather, says Proclus, ARITHMETIC IS MORE EXACT THEN GEOMETRY.<sup>8</sup> To an accurate calculator, if the diameter is set to one unit, the circumference of the inscribed dodecagon will be the side of the binomial [i.e. square root of the difference]  $72 - \sqrt{3888}$ . Whosoever declares any other result, will be mistaken, either the geometer in his measurements or the calculator in his numbers.”* (Vieta)

<sup>7</sup>A recent self-contained proof is given by H. Tsumura in the May 2004, *MAA Monthly*, 430–431.

<sup>8</sup>This phrase was written in Greek.



This fluent rendition is due to Marinus Taisbak, and the full text is worth reading. It certainly underlines how influential an algebraist and geometer Vieta was. Vieta, who was the first to introduce literals ('x' and 'y') into algebra, nonetheless rejected the use of negative numbers.

## 5 Pi's Adult Life with Calculus

In the later 17th century, Newton and Leibniz founded the calculus, and this powerful tool was quickly exploited to find new formulae for  $\pi$ . One early calculus-based formula comes from the integral:

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + \dots) dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$$

Substituting  $x = 1$  *formally* proves the well-known *Gregory-Leibniz formula* (1671–74)

$$(9) \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

James Gregory (1638–75) was the greatest of a large Scottish mathematical family. The point,  $x = 1$ , however, is on the boundary of the interval of convergence of the series. Justifying substitution requires a careful error estimate for the remainder or Lebesgue's monotone convergence theorem, etc., but most introductory texts ignore the issue.

**A Curious Anomaly in the Gregory Series.** In 1988, it was observed that Gregory's series for  $\pi$ ,

$$(10) \quad \pi = 4 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \right)$$

when truncated to 5,000,000 terms, differs strangely from the true value of  $\pi$ :

3.14159245358979323846464338327950278419716939938730582097494182230781640...	3.14159265358979323846264338327950288419716939937510582097494459230781640...		
2	-2	10	-122
			2770

Values differ as expected from truncating an alternating series, in the seventh place—a “4” which should be a “6.” But the next 13 digits are correct, and after another blip, for 12 digits. Of the first 46 digits, only four differ from the corresponding digits of  $\pi$ . Further, the “error” digits seemingly occur with a period of 14, as shown above. Such anomalous behavior begs explanation. A great place to start is by using Neil Sloane's Internet-based integer sequence recognition tool, available at [www.research.att.com/~njas/sequences](http://www.research.att.com/~njas/sequences). This tool has no difficulty recognizing the sequence of errors as twice *Euler numbers*. Even Euler numbers are generated by  $\sec x = \sum_{k=0}^{\infty} (-1)^k E_{2k} x^{2k} / (2k)!$ . The first few are 1, -1, 5, -61, 1385, -50521, 2702765. This discovery led to the following asymptotic expansion:

$$(11) \quad \frac{\pi}{2} - 2 \sum_{k=1}^{N/2} \frac{(-1)^{k+1}}{2k-1} \approx \sum_{m=0}^{\infty} \frac{E_{2m}}{N^{2m+1}}.$$

Now the genesis of the anomaly is clear: by chance the series had been truncated at 5,000,000 terms—exactly one-half of a fairly large power of ten. Indeed, setting  $N = 10,000,000$  in Equation (11) shows that the first hundred or so digits of the truncated series value are small perturbations of the correct decimal expansion for  $\pi$ . And the asymptotic expansions show up on the computer screen, as we observed above. On a hex computer with  $N = 16^7$  the corresponding strings are:

3.243F6A8885A308D313198A2E03707344A4093822299F31D0082EFA98EC4E6C89452821E...	3.243F6A6885A308D31319AA2E03707344A3693822299F31D7A82EFA98EC4DBF69452821E...		
2	-2	A	-7A
			2AD2

Name	Year	Correct Digits
Sharp (and Halley)	1699	71
Machin	1706	100
Strassnitzky and Dase	1844	200
Rutherford	1853	440
Shanks	1874	(707) 527
Ferguson ( <b>Calculator</b> )	1947	808
Reitwiesner et al. ( <b>ENIAC</b> )	1949	2,037
Genuys	1958	10,000
Shanks and Wrench	1961	100,265
Guilloud and Bouyer	1973	1,001,250

Figure 8: Calculus  $\pi$  Calculations

with the first being the correct value of  $\pi$ .

Similar phenomena occur for other constants. (See [2].) Also, knowing the errors means we can correct them and use (11) to make Gregory’s formula computationally tractable, despite the following discussion!

## 6 Calculus Era $\pi$ Calculations

Used naively, the beautiful formula (9) is computationally useless—so slow that hundreds of terms are needed to compute two digits. Sharp, under the direction of Halley<sup>9</sup>, see Figure 8, actually used  $\tan^{-1}(1/\sqrt{3})$  which is geometrically convergent.

Moreover, Euler’s (1738) trigonometric identity

$$(12) \quad \tan^{-1}(1) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$

produces a geometrically convergent rational series

$$(13) \quad \frac{\pi}{4} = \frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots + \frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \cdots$$

An even faster formula, found earlier by John Machin, lies similarly in the identity

$$(14) \quad \frac{\pi}{4} = 4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

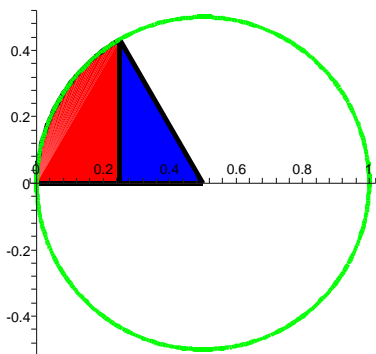
This was used in numerous computations of  $\pi$ , given in Figure 8, starting in 1706 and culminating with Shanks’ famous computation of  $\pi$  to 707 decimal digits accuracy in 1873 (although it was *found in 1945 to be wrong* after the 527-th decimal place, by Ferguson, during the last adding machine-assisted pre-computer computations.<sup>10</sup>).

**Newton’s arcsin computation.** Newton discovered a different more effective—actually a disguised arcsin—formula. He considering the area  $A$  of the left-most region shown in Figure 9. Now,  $A$  is the integral

$$(15) \quad A = \int_0^{1/4} \sqrt{x - x^2} dx.$$

<sup>9</sup>The astronomer and mathematician who largely built the Greenwich Observatory and after whom the comet is named.

<sup>10</sup>This must be some sort a record for the length of time needed to detect a mathematical error.



*"I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."*  
(Issac Newton, 1666)

The great fire of London ended the plague year in September 1666.

Figure 9: **Newton's method for  $\pi$**

Also,  $A$  is the area of the circular sector,  $\pi/24$ , less the area of the triangle,  $\sqrt{3}/32$ . Newton used his newly developed *binomial theorem* in (15):

$$\begin{aligned} A &= \int_0^{\frac{1}{4}} x^{1/2}(1-x)^{1/2} dx = \int_0^{\frac{1}{4}} x^{1/2} \left( 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \dots \right) dx \\ &= \int_0^{\frac{1}{4}} \left( x^{1/2} - \frac{x^{3/2}}{2} - \frac{x^{5/2}}{8} - \frac{x^{7/2}}{16} - \frac{5x^{9/2}}{128} \dots \right) dx \end{aligned}$$

Integrate term-by-term and combining the above produces

$$\pi = \frac{3\sqrt{3}}{4} + 24 \left( \frac{1}{3 \cdot 8} - \frac{1}{5 \cdot 32} - \frac{1}{7 \cdot 128} - \frac{1}{9 \cdot 512} \dots \right).$$

Newton used this formula to compute 15 digits of  $\pi$ . As noted, he later ‘apologized’ for “having no other business at the time.” A standard chronology [2, p. 294] says “*Newton significantly never gave a value for  $\pi$ .*” Caveat emptor all users of secondary sources.

**The Viennese computer.** Until around 1950 a computer was a person. This one, Johan Zacharias Dase (1824–1861) would demonstrate his extraordinary computational skill by, for example, multiplying

$$79532853 \times 93758479 = 7456879327810587$$

in 54 seconds; two 20-digit numbers in six minutes; two 40-digit numbers in 40 minutes; two 100-digit numbers in 8 hours and 45 minutes. In 1844, after being shown

$$\frac{\pi}{4} = \tan^{-1} \left( \frac{1}{2} \right) + \tan^{-1} \left( \frac{1}{5} \right) + \tan^{-1} \left( \frac{1}{8} \right)$$

he calculated  $\pi$  to 200 places *in his head* in two months, completing correctly—to my mind—the greatest mental computation ever. Dase later calculated a seven-digit logarithm table, and extended a table of integer factorizations to 10,000,000. Gauss requested that Dase be permitted to assist him, but Dase died shortly afterwards.

An amusing Machin-type identity, that is expressing Pi as linear a combination of arctan’s, due to the Oxford logician Charles Dodgson<sup>11</sup> is

$$\tan^{-1} \left( \frac{1}{p} \right) = \tan^{-1} \left( \frac{1}{p+q} \right) + \tan^{-1} \left( \frac{1}{p+r} \right),$$

valid whenever  $1 + p^2$  factors as  $qr$ .

<sup>11</sup>Dodgson is better known as Lewis Carroll, the author of *Alice in Wonderland*.

## 7 The Irrationality and Transcendence of $\pi$

One motivation for computations of  $\pi$  was very much in the spirit of modern experimental mathematics: to see if the decimal expansion of  $\pi$  repeats, which would mean that  $\pi$  is the ratio of two integers (i.e., rational), or to recognize  $\pi$  as *algebraic*—the root of a polynomial with integer coefficients—and later to look at digit distribution. The question of the *rationality* of  $\pi$  was settled in the late 1700s, when Lambert and Legendre proved (using continued fractions) that the constant is irrational.

The question of whether  $\pi$  was algebraic was settled in 1882, when Lindemann proved that  $\pi$  is *transcendental*. Lindemann's proof also settled, once and for all, the ancient Greek question of whether the circle could be squared with straight-edge and compass. It cannot be, because numbers that are the lengths of lines that can be constructed using ruler and compasses (often called *constructible numbers*) are necessarily algebraic, and squaring the circle is equivalent to constructing the value  $\pi$ .

But Aristophanes knew this and perhaps derided ‘circle-squarers’ ( $\tau\epsilon\tau\rho\alpha\gamma\omega\sigma\iota\epsilon\iota\nu$ ) in his play *The Birds* of 414 BCE. Likewise, the French Academy had stopped accepting proofs of the three great constructions of antiquity—squaring the circle, doubling the cube and trisecting the angle—centuries earlier.

We next give, in extenso, Ivan Niven's 1947 short proof of the irrationality of  $\pi$ . It well illustrates the ingredients of more difficult later proofs of irrationality of other constants, and indeed of Lindemann's proof of the transcendence of  $\pi$  building on Hermite's 1873 proof of the transcendence of  $e$ .

## 8 A Proof that $\pi$ is Irrational

**Proof.** Let  $\pi = a/b$ , the quotient of positive integers. We define the polynomials

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

the positive integer being specified later. Since  $n!f(x)$  has integral coefficients and terms in  $x$  of degree not less than  $n$ ,  $f(x)$  and its derivatives  $f^{(j)}(x)$  have integral values for  $x = 0$ ; also for  $x = \pi = a/b$ , since  $f(x) = f(a/b - x)$ . By elementary calculus we have

$$\frac{d}{dx}\{F'(x)\sin x - F(x)\cos x\} = F''(x)\sin x + F(x)\sin x = f(x)\sin x$$

and

$$\begin{aligned} \int_0^\pi f(x)\sin x dx &= [F'(x)\sin x - F(x)\cos x]_0^\pi \\ (16) \qquad \qquad \qquad &= F(\pi) + F(0). \end{aligned}$$

Now  $F(\pi) + F(0)$  is an *integer*, since  $f^{(j)}(0)$  and  $f^{(j)}(\pi)$  are integers. But for  $0 < x < \pi$ ,

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!},$$

so that the integral in (16) is *positive but arbitrarily small* for  $n$  sufficiently large. Thus (16) is false, and so is our assumption that  $\pi$  is rational. **QED**

**Irrationality measures.** We end this section by touching on the matter of *measures of irrationality*. The infimum  $\mu(\alpha)$  of those  $\mu > 0$  for which

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{q^\mu}$$



Figure 10: The ENIAC standing in the Smithsonian

for all integers  $p, q$  with sufficiently large  $q$ , is called the *Liouville-Roth constant* for  $\alpha$  and we say that we have an irrationality measure for  $\alpha$  if  $\mu(\alpha) < \infty$ .

Irrationality measures are difficult. Roth's theorem, [5], implies that  $\mu(\alpha) = 2$  for all algebraic irrationals, as is the case for almost all reals. Clearly,  $\mu(\alpha) = 1$  for rational  $\alpha$  and  $\mu(\alpha) = \infty$  iff and only if  $\alpha$  is Liouville numbers such as  $\sum 1/10^{n!}$ . It is known that  $\mu(e) = 2$  while in 1993 Hata showed that  $\mu(\pi) \leq 8.02$ . Similarly, it is known that  $\mu(\zeta(2)) \leq 5.45, \mu(\zeta(3)) \leq 4.8$  and  $\mu(\log 2) \leq 3.9$ .

A consequence of the existence of an irrationality measure  $\mu$  for  $\pi$ , is the ability to estimate quantities such as  $\limsup |\sin(n)|^{1/n} = 1$  for integer  $n$ , since for large integer  $m$  and  $n$  with  $m/n \rightarrow \pi$ , we have eventually

$$|\sin(n)| = |\sin(m\pi) - \sin(n)| \geq \frac{1}{2} |m\pi - n| \geq \frac{1}{2m^{\mu-1}}.$$

## 9 Pi in the Digital Age

With the substantial development of computer technology in the 1950s,  $\pi$  was computed to thousands and then millions of digits. These computations were greatly facilitated by the discovery soon after of advanced algorithms for the underlying high-precision arithmetic operations. For example, in 1965 it was found that the newly-discovered *fast Fourier transform* (FFT) [5, 7] could be used to perform high-precision multiplications much more rapidly than conventional schemes.

Such methods (e.g., for  $\div, \sqrt{x}$  see [5, 6, 7]) dramatically lowered the time required for computing  $\pi$  and other constants to high precision. We are now able to compute algebraic values of algebraic functions essentially as fast as we can multiply,  $O_B(M(N))$ . In spite of these advances, into the 1970s all computer evaluations of  $\pi$  still employed classical formulae, usually of Machin-type, see Figure 8. We will see below methods that compute  $N$  digits of  $\pi$  with time complexity  $O_B(M(N)) \log O_B(M(N))$ . Showing that the log term is unavoidable, as seems likely, would provide an algorithmic proof that  $\pi$  is not algebraic.

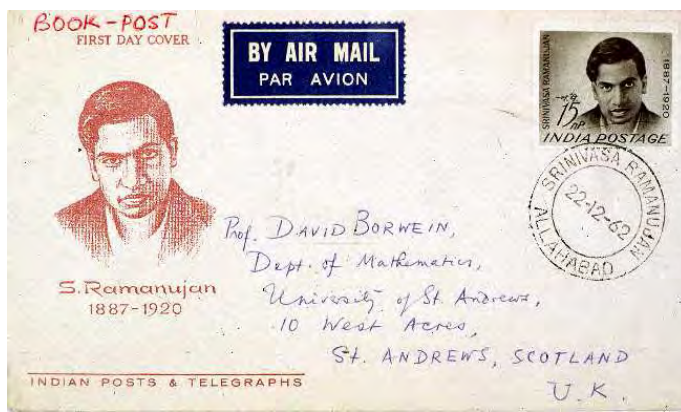


Figure 11: Ramanujan's seventy-fifth birthday stamp

G.N. Watson on viewing formulae of Ramanujan, such as (17), elegantly describes feeling

*"a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing "Day", "Night", "Evening", and "Dawn" which Michelangelo has set over the tomb of Giuliano de' Medici and Lorenzo de' Medici"*

**ENIAC: Electronic Numerical Integrator and Calculator.** Figure 10 shows ENIAC—a behemoth with a tiny brain from today's vantage point. Built in Aberdeen Maryland by the US Army, the Smithsonian's 20Gb image file reproduced here would have required thousands of ENIACs to store.

**Size/weight.** ENIAC had 18,000 vacuum tubes, 6,000 switches, 10,000 capacitors, 70,000 resistors, 1,500 relays, was 10 feet tall, occupied 1,800 square feet and weighed 30 tons. One gets a sense of its scale from Figure 10, which shows a modern monitor in front of it.

**Speed/memory.** A 1.5GHz Pentium does 3 million adds/sec. ENIAC did 5,000, three orders faster than any earlier machine. The first stored-memory computer, ENIAC could hold 200 digits.

**Input/output.** Data flowed from one accumulator to the next, and after each accumulator finished a calculation, it communicated its results to the next in line. The accumulators were connected to each other manually. The 1949 computation of  $\pi$  to 2,037 places on ENIAC took 70 hours in which output had to be constantly reintroduced as input.

**Ballantine's (1939) Series for  $\pi$ .** Another formula of Euler for arccot is

$$x \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n+1)! (x^2+1)^{n+1}} = \arctan\left(\frac{1}{x}\right).$$

This, intriguingly and usefully, allowed Guilloud and Boyer to reexpress the formula, used by them in 1973 to compute a million digits of Pi, viz,  $\pi/4 = 12 \arctan(1/18) + 8 \arctan(1/57) - 5 \arctan(1/239)$  in the efficient form

$$\pi = 864 \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n+1)! 325^{n+1}} + 1824 \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n+1)! 3250^{n+1}} - 20 \arctan\left(\frac{1}{239}\right),$$

where the terms of the second series are now just decimal shifts of the first.

**Ramanujan-type elliptic series.** Truly new types of infinite series formulae, based on elliptic integral approximations, were discovered by Srinivasa Ramanujan (1887–1920), shown in Figure 11, around 1910, but were not well known (nor fully proven) until quite recently when his writings were widely published. They are based on elliptic functions and are described at length in [2, 5, 7].

One of these series is the remarkable:

$$(17) \quad \frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)! (1103 + 26390k)}{(k!)^4 396^{4k}}.$$



Name	Year	Correct Digits
Miyoshi and Kanada	1981	2,000,036
Kanada-Yoshino-Tamura	1982	16,777,206
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada et. al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Kanada and Takahashi	Oct. 1995	6,442,450,938
Kanada and Takahashi	Jul. 1997	51,539,600,000
Kanada and Takahashi	Sep. 1999	206,158,430,000
Kanada-Ushiro-Kuroda	Dec. 2002	1,241,100,000,000

Figure 12: Post-calculus  $\pi$  Calculations

Each term of this series produces an additional *eight* correct digits in the result. When Gosper used this formula to compute 17 million digits of  $\pi$  in 1985, and it agreed to many millions of places with the prior estimates, *this concluded the first proof of (17)*, as described in [4]! Actually, Gosper first computed the simple continued fraction for  $\pi$ , hoping to discover some new things in its expansion, but found none.

At about the same time, David and Gregory Chudnovsky found the following rational variation of Ramanujan's formula<sup>12</sup>:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k+3/2}}$$

Each term of this series produces an additional 14 correct digits. The Chudnovskys implemented this formula using a clever scheme that enabled them to use the results of an initial level of precision to extend the calculation to even higher precision. They used this in several large calculations of  $\pi$ , culminating with a **then record computation** to over four billion decimal digits in 1994. Their remarkable story was compellingly told by Richard Preston in a prizewinning New Yorker article “The Mountains of Pi” (March 2, 1992).

While the Ramanujan and Chudnovsky series are in practice considerably more efficient than classical formulae, they share the property that the number of terms needed increases linearly with the number of digits desired: *if you want to compute twice as many digits of  $\pi$ , you must evaluate twice as many terms* of the series.

Relatedly, the Ramanujan-type series

$$(18) \quad \frac{1}{\pi} = \sum_{n=0}^{\infty} \left( \frac{\binom{2n}{n}}{16^n} \right)^3 \frac{42n+5}{16}.$$

allows one to compute the billionth binary digit of  $1/\pi$ , or the like, *without computing the first half* of the series, and is a foretaste of our later discussion of BBP formulae.

<sup>12</sup>It exists because  $\sqrt{-163}$  corresponds to an imaginary quadratic field with class number one

## 10 Reduced Operational Complexity Algorithms

In 1976, Eugene Salamin and Richard Brent independently discovered a *reduced complexity* algorithm for  $\pi$ . It is based on the **arithmetic-geometric mean iteration** (AGM) and some other ideas due to Gauss and Legendre around 1800, although Gauss, nor many after him, never directly saw the connection to effectively computing  $\pi$ .

**Quadratic Algorithm (Salamin-Brent).** Set  $a_0 = 1, b_0 = 1/\sqrt{2}$  and  $s_0 = 1/2$ . Calculate

$$(19) \quad a_k = \frac{a_{k-1} + b_{k-1}}{2} \quad (A) \quad b_k = \sqrt{a_{k-1}b_{k-1}} \quad (G)$$

$$(20) \quad c_k = a_k^2 - b_k^2, \quad s_k = s_{k-1} - 2^k c_k \quad \text{and compute} \quad p_k = \frac{2a_k^2}{s_k}.$$

Then  $p_k$  converges *quadratically* to  $\pi$ . Note the similarity between the arithmetic-geometric mean iteration (19), (which for general initial values converges fast to a non-elementary limit) and the out-of-kilter harmonic-geometric mean iteration (5) (which in general converges slowly to an elementary limit), and which is an arithmetic-geometric iteration in the reciprocals (see [5]).

Each iteration of the algorithm *doubles* the correct digits. Successive iterations produce 1, 4, 9, 20, 42, 85, 173, 347 and 697 good decimal digits of  $\pi$ , and takes  $\log N$  operations for  $N$  digits. Twenty-five iterations computes  $\pi$  to over 45 million decimal digit accuracy. A disadvantage is that each of these iterations must be performed to the precision of the final result.

In 1985, my brother Peter and I discovered families of algorithms of this type. For example, here is a genuinely third-order iteration:

**Cubic Algorithm.** Set  $a_0 = 1/3$  and  $s_0 = (\sqrt{3} - 1)/2$ . Iterate

$$r_{k+1} = \frac{3}{1 + 2(1 - s_k^3)^{1/3}}, \quad s_{k+1} = \frac{r_{k+1} - 1}{2} \quad \text{and} \quad a_{k+1} = r_{k+1}^2 a_k - 3^k (r_{k+1}^2 - 1).$$

Then  $1/a_k$  converges *cubically* to  $\pi$ . Each iteration *triples* the number of correct digits.

**Quartic Algorithm.** Set  $a_0 = 6 - 4\sqrt{2}$  and  $y_0 = \sqrt{2} - 1$ . Iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}} \quad \text{and} \quad a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2).$$

Then  $1/a_k$  converges *quartically* to  $\pi$ .<sup>13</sup>

To illustrate the stunning complexity reduction, let us write a complete set of algebraic equations approximating  $\pi$  to over a trillion digits.

The number  $\pi$  is transcendental and the number  $1/a_{20}$  computed next is algebraic  
nonetheless they coincide for over 1.5 trillion places.

Set  $a_0 = 6 - 4\sqrt{2}$ ,  $y_0 = \sqrt{2} - 1$  and then solve the following system:

---

<sup>13</sup>Note that only the power of 2 or 3 used in  $a_k$  depends on  $k$ .

## Mnemonics for Pi

*“Now I , even I, would celebrate  
In rhyme inapt, the great  
Immortal Syracusan, rivaled nevermore,  
Who in his wondrous lore,  
Passed on before  
Left men for guidance  
How to circles mensurate.” (30)*

*“How I want a drink, alcoholic of course,  
after the heavy lectures involving quantum  
mechanics.” (15)*

*“See I have a rhyme assisting my feeble  
brain its tasks ofttimes resisting.” (13)*

There are many more and longer mnemonics  
given in [2, p. 405, p.560, p. 659]



Figure 13: Yasumasa Kanada in his Tokyo office

$$\begin{aligned}
 y_1 &= \frac{1 - \sqrt[4]{1 - y_0^4}}{1 + \sqrt[4]{1 - y_0^4}}, a_1 = a_0 (1 + y_1)^4 - 2^3 y_1 (1 + y_1 + y_1^2) & y_{11} &= \frac{1 - \sqrt[4]{1 - y_{10}^4}}{1 + \sqrt[4]{1 - y_{10}^4}}, a_{11} = a_{10} (1 + y_{11})^4 - 2^{23} y_{11} (1 + y_{11} + y_{11}^2) \\
 y_2 &= \frac{1 - \sqrt[4]{1 - y_1^4}}{1 + \sqrt[4]{1 - y_1^4}}, a_2 = a_1 (1 + y_2)^4 - 2^5 y_2 (1 + y_2 + y_2^2) & y_{12} &= \frac{1 - \sqrt[4]{1 - y_{11}^4}}{1 + \sqrt[4]{1 - y_{11}^4}}, a_{12} = a_{11} (1 + y_{12})^4 - 2^{25} y_{12} (1 + y_{12} + y_{12}^2) \\
 y_3 &= \frac{1 - \sqrt[4]{1 - y_2^4}}{1 + \sqrt[4]{1 - y_2^4}}, a_3 = a_2 (1 + y_3)^4 - 2^7 y_3 (1 + y_3 + y_3^2) & y_{13} &= \frac{1 - \sqrt[4]{1 - y_{12}^4}}{1 + \sqrt[4]{1 - y_{12}^4}}, a_{13} = a_{12} (1 + y_{13})^4 - 2^{27} y_{13} (1 + y_{13} + y_{13}^2) \\
 y_4 &= \frac{1 - \sqrt[4]{1 - y_3^4}}{1 + \sqrt[4]{1 - y_3^4}}, a_4 = a_3 (1 + y_4)^4 - 2^9 y_4 (1 + y_4 + y_4^2) & y_{14} &= \frac{1 - \sqrt[4]{1 - y_{13}^4}}{1 + \sqrt[4]{1 - y_{13}^4}}, a_{14} = a_{13} (1 + y_{14})^4 - 2^{29} y_{14} (1 + y_{14} + y_{14}^2) \\
 y_5 &= \frac{1 - \sqrt[4]{1 - y_4^4}}{1 + \sqrt[4]{1 - y_4^4}}, a_5 = a_4 (1 + y_5)^4 - 2^{11} y_5 (1 + y_5 + y_5^2) & y_{15} &= \frac{1 - \sqrt[4]{1 - y_{14}^4}}{1 + \sqrt[4]{1 - y_{14}^4}}, a_{15} = a_{14} (1 + y_{15})^4 - 2^{31} y_{15} (1 + y_{15} + y_{15}^2) \\
 y_6 &= \frac{1 - \sqrt[4]{1 - y_5^4}}{1 + \sqrt[4]{1 - y_5^4}}, a_6 = a_5 (1 + y_6)^4 - 2^{13} y_6 (1 + y_6 + y_6^2) & y_{16} &= \frac{1 - \sqrt[4]{1 - y_{15}^4}}{1 + \sqrt[4]{1 - y_{15}^4}}, a_{16} = a_{15} (1 + y_{16})^4 - 2^{33} y_{16} (1 + y_{16} + y_{16}^2) \\
 y_7 &= \frac{1 - \sqrt[4]{1 - y_6^4}}{1 + \sqrt[4]{1 - y_6^4}}, a_7 = a_6 (1 + y_7)^4 - 2^{15} y_7 (1 + y_7 + y_7^2) & y_{17} &= \frac{1 - \sqrt[4]{1 - y_{16}^4}}{1 + \sqrt[4]{1 - y_{16}^4}}, a_{17} = a_{16} (1 + y_{17})^4 - 2^{35} y_{17} (1 + y_{17} + y_{17}^2) \\
 y_8 &= \frac{1 - \sqrt[4]{1 - y_7^4}}{1 + \sqrt[4]{1 - y_7^4}}, a_8 = a_7 (1 + y_8)^4 - 2^{17} y_8 (1 + y_8 + y_8^2) & y_{18} &= \frac{1 - \sqrt[4]{1 - y_{17}^4}}{1 + \sqrt[4]{1 - y_{17}^4}}, a_{18} = a_{17} (1 + y_{18})^4 - 2^{37} y_{18} (1 + y_{18} + y_{18}^2) \\
 y_9 &= \frac{1 - \sqrt[4]{1 - y_8^4}}{1 + \sqrt[4]{1 - y_8^4}}, a_9 = a_8 (1 + y_9)^4 - 2^{19} y_9 (1 + y_9 + y_9^2) & y_{19} &= \frac{1 - \sqrt[4]{1 - y_{18}^4}}{1 + \sqrt[4]{1 - y_{18}^4}}, a_{19} = a_{18} (1 + y_{19})^4 - 2^{39} y_{19} (1 + y_{19} + y_{19}^2) \\
 y_{10} &= \frac{1 - \sqrt[4]{1 - y_9^4}}{1 + \sqrt[4]{1 - y_9^4}}, a_{10} = a_9 (1 + y_{10})^4 - 2^{21} y_{10} (1 + y_{10} + y_{10}^2) & y_{20} &= \frac{1 - \sqrt[4]{1 - y_{19}^4}}{1 + \sqrt[4]{1 - y_{19}^4}}, a_{20} = a_{19} (1 + y_{20})^4 - 2^{41} y_{20} (1 + y_{20} + y_{20}^2).
 \end{aligned}$$

This quartic algorithm, with the Salamin–Brent scheme, was first used by Bailey, see Figure 14, in 1986 and was used repeatedly by Yasumasa Kanada, see Figure 13, in Tokyo in computations of  $\pi$  over the past 15 years or so, culminating in a 200 billion decimal digit computation in 1999, see Figure 12. Only 35 years earlier in 1963, Dan Shanks—a very knowledgeable participant—was confident that computing a billion digits was forever impossible. Today it is easy on a modest laptop.



Figure 14: David Bailey on a Berkeley Bus

**Philosophy of mathematics.** In 1997 the first occurrence of the sequence 0123456789 was found (later than expected) in the decimal expansion of  $\pi$  starting at the 17,387,594,880-th digit after the decimal point. In consequence the status of several famous *intuitionistic examples* due to Brouwer and Heyting has changed. These challenge the *principle of the excluded middle*—either a predicate holds or it does not—and involve classically well-defined objects that for an intuitionist are ill-founded until one could determine when or if the sequence occurred.<sup>14</sup>

## 11 Back to the Future

In December 2002, Kanada computed  $\pi$  to over **1.24 trillion decimal digits**. His team first computed  $\pi$  in hexadecimal (base 16) to 1,030,700,000,000 places, using the following two arctangent relations:

$$\pi = 48 \tan^{-1} \frac{1}{49} + 128 \tan^{-1} \frac{1}{57} - 20 \tan^{-1} \frac{1}{239} + 48 \tan^{-1} \frac{1}{110443}$$

$$\pi = 176 \tan^{-1} \frac{1}{57} + 28 \tan^{-1} \frac{1}{239} - 48 \tan^{-1} \frac{1}{682} + 96 \tan^{-1} \frac{1}{12943}.$$

The first formula was found in 1982 by K. Takano, a high school teacher and song writer. The second formula was found by F. C. W. Störmer in 1896. Kanada verified the results of these two computations agreed, and then converted the hex digit sequence to decimal. The resulting decimal expansion was checked by converting it back to hex. These conversions are themselves non-trivial, requiring massive computation.

This process is quite different from those of the previous quarter century. One reason is that reduced operational complexity algorithms, require full-scale multiply, divide and square root operations. These in turn require large-scale FFT operations, which demand huge amounts of memory, and massive all-to-all communication between nodes of a large parallel system. For this latest computation, even the very large system available in Tokyo did not have sufficient memory and network bandwidth to perform these operations at reasonable efficiency levels—at least not for trillion-digit computations. Utilizing arctans again meant using many more arithmetic operations, but no system-scale FFTs, and it can be implemented using  $\times, \div$  by smallish integer values—additionally, hex is somewhat more efficient!

Kanada and his team evaluated these two formulae using a scheme analogous to that employed by Gosper and by the Chudnovskys in their series computations, in that they were able to avoid explicitly storing the multiprecision numbers involved. This resulted in a scheme that is roughly competitive in *numerical* efficiency with the Salamin-Brent and Borwein quartic algorithms they had previously used, but with a significantly lower total memory requirement. Kanada used a 1 Tbyte main memory system, as with the previous computation, yet got six times as many digits. Hex and decimal evaluations included, it ran 600 hours on a 64-node Hitachi, with the main segment of the program running at a sustained rate of nearly 1 Tflop/sec.

<sup>14</sup>See J.M. Borwein, “Brouwer-Heyting sequences converge,” *Mathematical Intelligencer*, **20** (1998), 14-15.

		Hex Digit	Occurrences
		<hr/>	
		0	62499881108
		1	62500212206
		2	62499924780
		3	62500188844
		4	62499807368
		5	62500007205
		6	62499925426
		7	62499878794
		8	<u>62500216752</u>
		9	62500120671
		A	62500266095
		B	62499955595
		C	62500188610
		D	62499613666
		E	62499875079
		F	62499937801
		<hr/>	
		Total	1000000000000
		<hr/>	
Decimal Digit	Occurrences		
0	99999485134		
1	99999945664		
2	100000480057		
3	99999787805		
4	<u>100000357857</u>		
5	99999671008		
6	99999807503		
7	99999818723		
8	100000791469		
9	99999854780		
		<hr/>	
Total	1000000000000		

Figure 15: **Apparently random behaviour of  $\pi$  base 10 and 16**

## 12 Why Pi?

What possible motivation lies behind modern computations of  $\pi$ , given that questions such as the irrationality and transcendence of  $\pi$  were settled more than 100 years ago? One motivation is the raw challenge of harnessing the stupendous power of modern computer systems. Programming such calculations are definitely not trivial, especially on large, distributed memory computer systems.

There have been substantial practical spin-offs. For example, some new techniques for performing the fast Fourier transform (FFT), heavily used in modern science and engineering computing, had their roots in attempts to accelerate computations of  $\pi$ . And always the computations help in road-testing computers—often uncovering subtle hardware and software errors.

Beyond practical considerations lies the abiding interest in the fundamental question of the *normality* (*digit randomness*) of  $\pi$ . Kanada, for example, has performed detailed statistical analysis of his results to see if there are any statistical abnormalities that suggest  $\pi$  is not normal, so far ‘no’, see Figures 15 and 16. Indeed the first computer computation of  $\pi$  and  $e$  on ENIAC was so motivated by John von Neumann. The digits of  $\pi$  have been studied more than any other single constant, in part because of the widespread fascination with and recognition of  $\pi$ . Kanada reports that the 10 decimal digits ending in position one trillion are 6680122702, while the 10 hexadecimal digits ending in position one trillion are 3F89341CD5.

**Changing world views.** In retrospect, we may wonder why in antiquity  $\pi$  was not *measured* to an accuracy in excess of 22/7? Perhaps it reflects not an inability to do so but a very different mind set to a modern experimental—Baconian or Popperian—one. In the same vein, one reason that Gauss and Ramanujan did not further develop the ideas in their identities for  $\pi$  is that an iterative algorithm, as opposed to explicit results, was not as satisfactory for them (especially Ramanujan). Ramanujan much preferred formulae like

$$\pi \approx \frac{3}{\sqrt{67}} \log(5280), \quad \frac{3}{\sqrt{163}} \log(640320) \approx \pi$$

correct to 9 and 15 decimal places both of which rely on deep number theory. Contrastingly, Ramanujan in his

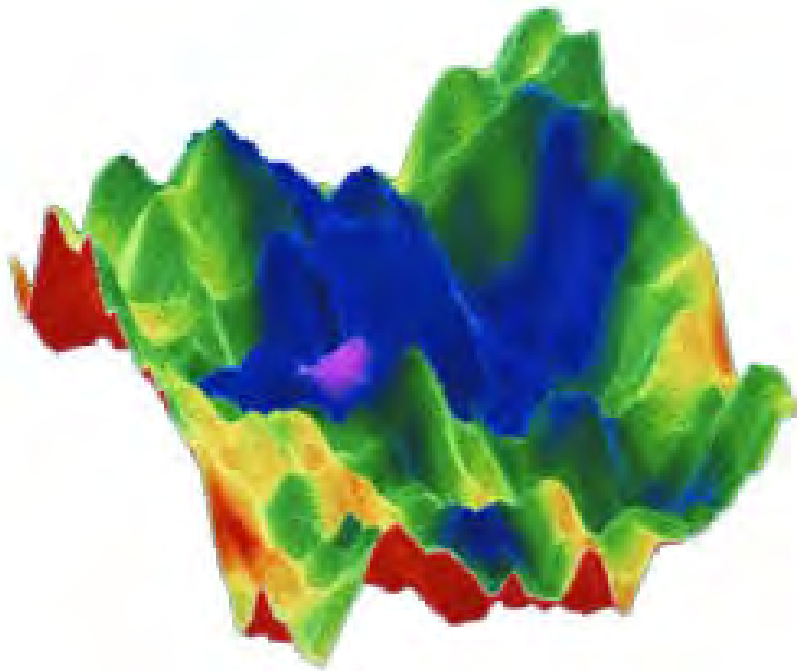


Figure 16: A random walk on the first one million digits of  $\pi$  (Courtesy D. and G. Chudnovsky)

famous 1914 paper *Modular Equations and Approximations to Pi* [2, p.253] found

$$\left(9^2 + \frac{19^2}{22}\right)^{1/4} = 3.14159265\bar{2}58\dots$$

“*empirically, and it has no connection with the preceding theory.*” Only the marked digit is wrong.

**Discovering the  $\pi$  Iterations.** The genesis of the  $\pi$  algorithms and related material is an illustrative example of experimental mathematics. My brother and I in the early eighties had a family of quadratic algorithms for  $\pi$ , [5], call them  $\mathcal{P}_N$ , of the kind we saw above. For  $N = 1, 2, 3, 4$  we could prove they were correct but and only conjectured for  $N = 5, 7$ . In each case the algorithm *appeared* to converge quadratically to  $\pi$ . On closer inspection while the provable cases were correct to 5,000 digits, the empirical versions of agreed with  $\pi$  to roughly 100 places only. Now in many ways to have discovered a “natural” number that agreed with  $\pi$  to that level—and no more—would have been more interesting than the alternative. That seemed unlikely but recoding and rerunning the iterations kept producing identical results.

Two decades ago even moderately high precision calculation was less accessible, and the code was being run remotely over a phone-line in a Berkeley Unix integer package. After about six weeks, it transpired that the package’s *square root algorithm was badly flawed, but only if run with an odd precision of more than sixty digits!* And for idiosyncratic reasons that had only been the case in the two unproven cases. Needless to say, tracing the bug was a salutary and somewhat chastening experience. And it highlights why one checks computations using different sub-routines and methods.

## 13 How to Compute the $N$ -th Digits of $\pi$

One might be forgiven for thinking that essentially everything of interest with regards to  $\pi$  has been dealt with. This is suggested in the closing chapters of Beckmann’s 1971 book *A History of  $\pi$* . Ironically, the Salamin–Brent quadratically convergent iteration was discovered only five years later, and the higher-order convergent





Figure 17: A Vancouver  $\pi$  Day poster

algorithms followed in the 1980s. Then in 1990, Rabinowitz and Wagon discovered a ‘spigot’ algorithm for  $\pi$ —the digits ‘drip out’ one by one. This permits successive digits of  $\pi$  (in any desired base) to be computed by a relatively simple recursive algorithm based on the *all previously* generated digits.

Even insiders are sometimes surprised by a new discovery. Prior to 1996, most folks thought if you want to determine the  $d$ -th digit of  $\pi$ , you had to generate the (order of) the entire first  $d$  digits. This is not true, at least for hex (base 16) or binary (base 2) digits of  $\pi$ . In 1996, Peter Borwein, Plouffe, and Bailey found an algorithm for computing individual hex digits of  $\pi$ . It (1) yields a modest-length hex or binary digit string for  $\pi$ , from an arbitrary position, using no prior bits; (2) is implementable on any modern computer; (3) requires no multiple precision software; (4) requires very little memory; and (5) has a computational cost growing only slightly faster than the digit position. For example, the millionth hexadecimal digit (four millionth binary digit) of  $\pi$  can be found in four seconds on a present generation Apple G5 workstation.

This new algorithm is not fundamentally faster than the best known schemes if used for computing *all* digits of  $\pi$  up to some position, but its elegance and simplicity are of considerable interest, and is easy to parallelize. It is based on the following at-the-time new formula for  $\pi$ :

$$(21) \quad \pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right)$$

which was discovered using *integer relation methods* (see [7]), with a computer program that ran for several months and then produced the (equivalent) relation:

$$\pi = 4F\left(1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4}\right) + 2 \tan^{-1}\left(\frac{1}{2}\right) - \log 5$$

where  $F(1, 1/4; 5/4, -1/4) = 0.955933837\dots$  is a Gaussian hypergeometric function.

**Proof of (21).**<sup>15</sup> For  $0 < k < 8$ ,

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8} dx = \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i} dx = \frac{1}{2^{k/2}} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}$$

Thus, one can write

$$\sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8} dx,$$

<sup>15</sup> *Maple* and *Mathematica* can now prove (21)

Position	Hex strings starting at this Position
$10^6$	26C65E52CB4593
$10^7$	17AF5863EFED8D
$10^8$	ECB840E21926EC
$10^9$	85895585A0428B
$10^{10}$	921C73C6838FB2
$10^{11}$	9C381872D27596
$1.25 \times 10^{12}$	07E45733CC790B
$2.5 \times 10^{14}$	E6216B069CB6C1



**Borweins and Plouffe (MSNBC, 1996)**

Figure 18: Percival's hexadecimal findings

which on substituting  $y = \sqrt{2x}$  becomes

$$\int_0^1 \frac{16y - 16}{y^4 - 2y^3 + 4y - 4} dy = \int_0^1 \frac{4y}{y^2 - 2} dy - \int_0^1 \frac{4y - 8}{y^2 - 2y + 2} dy = \pi.$$

We are done. **QED**

**The algorithm in action.** In 1997, Fabrice Bellard of INRIA computed 152 binary digits of  $\pi$  starting at the trillionth position. The computation took 12 days on 20 workstations working in parallel over the Internet. Bellard's scheme is based on the following variant of (21):

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k(2k+1)} - \frac{1}{64} \sum_{k=0}^{\infty} \frac{(-1)^k}{1024^k} \left( \frac{32}{4k+1} + \frac{8}{4k+2} + \frac{1}{4k+3} \right),$$

which permits hex or binary digits of  $\pi$  to be calculated roughly 43% faster than (21).

In 1998 Colin Percival, then a 17-year-old student at Simon Fraser University, utilized 25 machines to calculate first the five trillionth hexadecimal digit, and then the ten trillionth hex digit. In September, 2000, he found the quadrillionth binary digit is **0**, a computation that required *250 CPU-years, using 1734 machines in 56 countries*. We record some computational results in Figure 18.

A last comment for this section is that Kanada was able to confirm his 2002 computation in only 21 hours by computing a 20 hex digit string starting at the trillionth digit, and comparing this string to the hex string he had initially obtained in over 600 hours. Their agreement provided enormously strong confirmation.

## 14 Further BBP Digit Formulae

Motivated as above, constants  $\alpha$  of the form

$$(22) \quad \alpha = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)2^k},$$

where  $p(k)$  and  $q(k)$  are integer polynomials, are said to be in the class of *binary (Borwein-Bailey-Plouffe) BBP numbers*. I illustrate for  $\log 2$  why this permits one to calculate isolated digits in the binary expansion:

$$(23) \quad \log 2 = \sum_{k=0}^{\infty} \frac{1}{k2^k}.$$

We wish to compute a few binary digits beginning at position  $d+1$ . This is equivalent to calculating  $\{2^d \log 2\}$ , where  $\{\cdot\}$  denotes fractional part. We can write

$$(24) \quad \{2^d \log 2\} = \left\{ \left\{ \sum_{k=0}^d \frac{2^{d-k}}{k} \right\} + \left\{ \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k} \right\} \right\} = \left\{ \left\{ \sum_{k=0}^d \frac{2^{d-k} \bmod k}{k} \right\} + \left\{ \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k} \right\} \right\}.$$

The key observation is that the numerator of the first sum in (24),  $2^{d-k} \bmod k$ , can be calculated rapidly by *binary exponentiation*, performed modulo  $k$ . That is, it is economically performed by a factorization based on the binary expansion of the exponent. For example,

$$3^{17} = (((3^2)^2)^2)^2 \cdot 3$$

uses only five multiplications, not the usual 16. It is important to reduce each product modulo  $k$ . Thus,  $3^{17} \bmod 10$  is done as

$$3^2 = 9; 9^2 = 1; 1^2 = 1; 1^2 = 1; 1 \times 3 = 3.$$

A natural question in light of (21) is whether there is a formula of this type and an associated computational strategy to compute individual *decimal* digits of  $\pi$ . Searches conducted by numerous researchers have been unfruitful and recently D. Borwein (my father), Gallway and I have shown that there are no BBP formulae of the *Machin-type* (as defined in [7]) of (21) for  $\pi$  unless the base is a power of two [7].

**Ternary BBP formulae.** Yet, BBP formulae exist in other bases for some constants. For example, Broadhurst found this ternary BBP formula for  $\pi^2$ :

$$\begin{aligned} \pi^2 = & \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{3^{9k}} \left\{ \frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} \right. \\ & - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \left. \frac{1}{(12k+11)^2} \right\}, \end{aligned}$$

and  $\pi^2$  also has a binary BBP formula.

Also, the volume  $V_8$  in *hyperbolic space* of the *figure-eight knot complement* is well known to be

$$V_8 = 2\sqrt{3} \sum_{n=1}^{\infty} \frac{1}{n \binom{2n}{n}} \sum_{k=n}^{2n-1} \frac{1}{k} = 2.029883212819307250042405108549 \dots$$

Surprisingly, it is also expressible as

$$V_8 = \frac{\sqrt{3}}{9} \sum_{n=0}^{\infty} \frac{(-1)^n}{27^n} \left\{ \frac{18}{(6n+1)^2} - \frac{18}{(6n+2)^2} - \frac{24}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{2}{(6n+5)^2} \right\},$$

again discovered numerically by Broadhurst, and proved in [7]. A beautiful representation by Helaman Ferguson the mathematical sculptor is given in Figure 19. Ferguson produces art inspired by deep mathematics, but not by a formulaic approach.

**Normality and dynamics.** Finally, Bailey and Crandall in 2001 made exciting connections between the existence of a  $b$ -ary BBP formula for  $\alpha$  and its *normality* base  $b$  (uniform distribution of base- $b$  digits)<sup>16</sup>. They

<sup>16</sup>See [www.sciencenews.org/20010901/bob9.asp](http://www.sciencenews.org/20010901/bob9.asp).



These ‘subtractive’ acrylic circles represent the weights  $[4, -2, -2, -1]$  in Equation (21)



Figure 19: Ferguson’s “Eight-Fold Way” and his BBP acrylic circles

make a reasonable, hence very hard, conjecture about the *uniform distribution of a related chaotic dynamical system*. This conjecture implies: *Existence of a ‘BBP’ formula base  $b$  for  $\alpha$  ensures the normality base  $b$  of  $\alpha$* . For  $\log 2$ , illustratively<sup>17</sup>, the dynamical system, base 2, is to set  $x_0 = 0$  and compute

$$x_{n+1} \leftrightarrow 2 \left( x_n + \frac{1}{n} \right) \mod 1.$$

## 15 ... Life of Pi.

As we have seen the life of Pi captures a great deal of mathematics—algebraic, geometric and analytic, both pure and applied—along with some history and philosophy. It engages many of the greatest mathematicians and some quite interesting characters along the way. Among the saddest and least well understood episodes was an abortive 1896 attempt in Indiana to legislate the value of Pi. The bill, reproduced in [2, p. 231-235], is accurately described by Singmaster, [2, p. 236-239]. Much life remains in this most central of numbers.

At the end of the novel, Piscine (Pi) Molitor writes

*“I am a person who believes in form, in harmony of order. Where we can, we must give things a meaningful shape. For example—I wonder—could you tell my jumbled story in exactly one hundred chapters, not one more, not one less? I’ll tell you, that’s one thing I hate about my nickname, the way that number runs on forever. It’s important in life to conclude things properly. Only then can you let go.”*

We may well not share the sentiment, but we should celebrate that Pi knows  $\pi$  to be irrational.

**Acknowledgements.** Thanks are due to many, especially my close collaborators P. Borwein and D. Bailey.

<sup>17</sup>In this case it is easy to use Weyl’s criterion for equidistribution to establish this equivalence without mention of BBP numbers.

3 . 1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679  
8214808651328230664709384460955058223172535940812848111745028410270193852110555964462294895493038196  
4428810975665933446128475648233786783165271201909145648566923460348610454326648213393607260249141273  
7245870066063155881748815209209628292540917153643678925903600113305305488204665213841469519415116094  
3305727036575959195309218611738193261179310511854807446237996274956735188575272489122793818301194912  
9833673362440656643086021394946395224737190702179860943702770539217176293176752384674818467669405132  
0005681271452635608277857713427577896091736371787214684409012249534301465495853710507922796892589235  
4201995611212902196086403441815981362977477130996051870721134999999837297804995105973173281609631859  
5024459455346908302642522308253344685035261931188171010003137838752886587533208381420617177669147303  
59825349042875546873115956286388235378759375195778185778053217122680661300192787661119590921642019893

Figure 20: 1,001 **Decimal Digits of Pi**

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There are many Internet resources on Pi, a reliable selection is kept at [www.expmath.info](http://www.expmath.info).